Dimensional Analysis

Dimensions of Quantities

The dimension of a quantity is used to describe what *sort* of quantity is being measured. Common quantities are mass, time, and length (or distance). Independent of the unit of measurement, the same quantities will have the same dimension; this allows for the consistency of equations to be checked. The dimension of a quantity is denoted by square brackets, for example, $[\text{seconds}] = T$. Mass and length are expressed as M and L respectively. Hours, days, weeks, and milliseconds are all different measurements of time - however, they all have the same dimension. Constants have no effect on dimensions and are said to be **dimensionless.**

To find the dimension of a quantity which is the product or quotient of other quantities, combine the dimensions of these quantities in the same way:

Example 1: Find the dimensions of **a.)** velocity, **b.)** acceleration, **c.)** density and **d.)** θ, an angle measured in radians.

a.) For an object moving in a straight line at a constant speed, the velocity is found by dividing the distance s that it travels by time t. Thus, $v = \frac{s}{t}$. Find the dimension of v by quotienting the dimensions of distance and time.

The dimensions of distance and time are

$$[s] = L, \qquad [t] = T$$

Thus, the dimensions for velocity are

$$[v] = \frac{[s]}{[t]} = \frac{L}{T} = LT^{-1}$$

b.) Acceleration is the rate of change of velocity, $a = \frac{dv}{dt}$. In this equation, dv is a 'small change' in v for a 'small change' in t and so $[ds] = [s]$, $[dt] = [t]$. This will allow the dimensions of a to be found in the same way as in part **a.)**.

$$a = \frac{dv}{dt}$$

$$\therefore [a] = \frac{[dv]}{[dt]} = \frac{[v]}{[t]} = \frac{LT^{-1}}{T} = LT^{-2}$$

c.) Density is defined as the ratio of mass to volume, $d = \frac{m}{v}$. Find the dimension of volume by using volume as the product of three lengths, and so its dimension will be the dimension of length cubed.

$$d = \frac{m}{v}$$

$$\therefore [d] = \frac{[m]}{[v]} = \frac{M}{[\text{length}]^3} = \frac{M}{L^3} = ML^{-3}$$

d.) Use the definition: angle in radians $= \frac{\text{arc length}}{\text{radius}}$ to find the dimension of an angle. As converting angles from radians to degrees requires the multiplication by a constant, angles in degrees are also dimensionless.

$$[\text{angle in radians}] = \frac{[\text{arc length}]}{[\text{radius}]} = \frac{L}{L} = 1$$

$\therefore$ Angles are dimensionless quantities. They have units, but no dimension.

When adding or subtracting quantities, they must have the same dimension. For example, seconds could not be subtracted from metres since they are measurements of different quantities. This principle is used to check whether a formula is dimensionally consistent. This is also called an error check.

Example 2: Given that a and b are measurements of time, c is a measurement of length, and d and e are measurement of mass, check that the following expressions are dimensionally consistent. If so, state their dimension. **a.)** $ac + bc$, **b.)** $\frac{ad}{c} + cd$, **c.)** $\frac{a^2}{ce} + \frac{b^2}{cd}$.

a.) Check that both terms have the same dimension. If so, combine them. The dimension of the sum is the same as the dimension of each term.

$$[ac] = [a][c] = TL$$

$$[bc] = [b][c] = TL$$

$\therefore [ac + bc] = TL$ is dimensionally consistent and has dimension TL.

b.) Find the dimension of both terms in the sum. Since these dimensions are not the same, the quantities cannot be added, and so this equation is not dimensionally consistent.

$$\left[\frac{ad}{c}\right] = \frac{[ad]}{[c]} = \frac{[a][d]}{[c]} = \frac{TM}{L} = TML^{-1}$$

$$[cd] = [c][d] = LM$$

$LM \neq TML^{-1}$ $\therefore$ This equation is dimensionally inconsistent.

c.) Again, check that both terms in the sum have the same dimension, and state that the equation is dimensionally consistent if so.

$$\left[\frac{a^2}{ce}\right] = \frac{[a]^2}{[c][e]} = \frac{T^2}{LM} = T^2L^{-1}M^{-1}$$

$$\left[\frac{b^2}{cd}\right] = \frac{[b]^2}{[c][d]} = \frac{T^2}{LM} = T^2L^{-1}M^{-1}$$

$\therefore \frac{a^2}{ce} + \frac{b^2}{cd}$ is dimensionally consistent, and has dimension $T^2L^{-1}M^{-1}$.

Example 3: Find the dimension of π in the formula for the surface area of a cylinder: $A = 2\pi r^2 + 2\pi rh$ where r is the radius of the cylinder and h is the height.

Rearrange the equation to find an expression for π in terms of the other variables. Then, equate the dimension of π to that of the right-hand side of the equation. Find the dimensions of each term in the expression, and then combine them using products and quotients where appropriate. Since the dimension of π is not an expression involving any quantity, it is said to be dimensionless.

$$A = 2\pi r(r + h) \Rightarrow \pi = \frac{A}{2r(r + h)}$$

$$\therefore [\pi] = \frac{[A]}{[2r][r + h]}$$

$$[A] = L^2$$

$$[2r] = L, \qquad [h] = L \therefore [r + h] = L$$

$$\therefore \frac{[A]}{[2r][r + h]} = \frac{L^2}{L \cdot L} = 1$$

$\therefore [\pi] = 1$, so π is a dimensionless constant.

Predicting Formulae

Dimensional consistency can be used to predict formulas for quantities by equating the dimensions on both sides of the given formula.

Example 4: The time, t, between two vibrations of a guitar string of length ℓ, mass m, and tension F is of the form $t = 2\pi F^\alpha m^\beta \ell^\gamma$, where 2π is a dimensionless constant. Find the values of α, β and γ and hence find a formula for t in terms of tension, mass, and length.

The dimension of t will be the same as the product of the dimensions of the quantities in the formula, all raised to their respective powers. Equate both sides of the equation as dimensions.

$$[t] = T = \left[2\pi F^\alpha m^\beta \ell^\gamma\right] = [2\pi][F^\alpha]\left[m^\beta\right][l^\gamma] = [F]^\alpha[m]^\beta[\ell]^\gamma$$

$$[F] = [ma] = [m][a] = MLT^{-2} \therefore [F]^\alpha = M^\alpha L^\alpha T^{-2\alpha}$$

$$[m] = M, [\ell] = L \Rightarrow [m]^\beta = M^\beta, [\ell]^\gamma = L^\gamma$$

$$\therefore \left[2\pi F^\alpha m^\beta l^\gamma\right] = M^\alpha L^\alpha T^{-2\alpha} M^\beta L^\gamma$$

$$T = M^{\alpha+\beta} L^{\alpha+\gamma} T^{-2\alpha}$$

The equation must be dimensionally consistent, so equate the left-hand side's dimension to that of the right. Then, equate each dimensions' power on the left- and right-hand side to form equations for each of the unknowns. Solve these to find each one and write the formula for t using the values of α, β and γ that have been found.

Coefficient of T:

$$\Rightarrow 1 = -2\alpha \Rightarrow \alpha = -\frac{1}{2}$$

Coefficient of L:

$$0 = \alpha + \gamma = -\frac{1}{2} + \gamma \Rightarrow \gamma = \frac{1}{2}$$

Coefficient of M:

$$0 = \alpha + \beta = -\frac{1}{2} + \beta \Rightarrow \beta = \frac{1}{2}$$

$$\therefore t = 2\pi F^{-\frac{1}{2}} m^{\frac{1}{2}} \ell^{\frac{1}{2}} = 2\pi\sqrt{\frac{m\ell}{F}}$$

Finding the Dimensions of Units

Dimensional consistency can be used to find the dimensions of derived SI units such as Joules, Newtons and Watts.

Example 5: Find the dimensions of Newton-seconds (unit of impulse) by using the definition: acceleration is the second derivative of distance with respect to time.

Write the equation for impulse and find the dimensions of each quantity in the equation. Since the second derivative is the first derivative 'per unit time', find the dimension of this by using that $\left[\frac{d}{dt}\right] = \frac{1}{[dt]} = \frac{1}{T}$. Combine the dimensions of each term in the formula to arrive at the result.

$$Impulse = force \cdot time$$

$$force = mass\,.\,acceleration$$

$$a = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \therefore [a] = \frac{1}{[dt]} \cdot \frac{[dx]}{[dt]} = \frac{L}{T^2} = LT^{-2}$$

$$\therefore [force] = [ma] = MLT^{-2}$$

$$\therefore [impulse] = [force][time] = MLT^{-2} \cdot T = MLT^{-1}$$

$\therefore$ The dimension of Newton Second is MLT^{-1}.